Energy conservation of a uniformly accelerated point charge

Chung-Sang Ng
Physics Department, Auburn University, Auburn, Alabama 36849
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Direct numerical calculation shows that the total electromagnetic-field energy of a one-dimensional uniformly accelerated classical point charge at one instant is the same as that at another instant when the charge is moving at the same speed but in the opposite direction of the first instant. This seems contradictory to the fact that the radiation power calculated by the Larmor formula is constant, but is required by the fact that the radiation reaction vanishes. It is also shown numerically that the electromagetic-field energy changes a finite amount when a charge begins or ends its uniformly accelerated motion. This change of energy is equal to the work done against a δ-function radiation reaction. The implications of these results on the question of whether a uniformly accelerated charge radiates are discussed.

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I. INTRODUCTION

Two facts in classical electrodynamics seem contradictory. On the one hand, the Larmor formula states that a charge radiates with a finite emission power whenever the charge is accelerating. On the other hand, the radiation reaction on a uniformly accelerated charge (UAC) vanishes. Does a UAC radiate? A lot of work has been done on this problem [1–12]. For a detailed history, we refer to Refs. [1–3]. Briefly speaking, some concluded that a UAC does not radiate [5–7], but most authors argued that it does [1–3, 8–12]. Among the latter, some explained energy conservation by some kind of "internal energy" or so-called "acceleration energy" of the charge [1,8,9,11]. Some others argued that this "acceleration energy" is actually the electromagnetic- (EM) field energy "near" the charge [1,2]. Still some others thought that the radiation energy actually comes from the δ-function field along the moving boundary that separates the region with the EM field from the region without [3,12]. These points of view are quite contradictory to one another.

If the explanation based on non-EM "internal energy" is correct, the total EM-field energy should increase at a rate given by the Larmor formula. In other words, the increase of the EM-field energy cannot be explained by the work done against the radiation reaction alone. To the best of our knowledge, no one has done a direct calculation of the total EM-field energy in order to see whether this is true or not. Moreover, although classical radiation-reaction theory [13] is well established, there are still problems such as "runaway solutions" or "violation of causality" within it [14]. Whether radiation-reaction theory implies energy conservation is not easily seen directly. The EM field of a UAC provides a direct test of this. Therefore, we will present such a calculation here.

What we mean by "direct" is that we compare the EM-field energy directly, without using a conservation law such as Poynting's theorem. There are two difficulties in doing so. First, if we consider the case that a charge is in uniformly accelerated motion all the time, then there is a δ-function field that brings an infinite contribution to the total EM-field energy. To avoid this, we may consider a charge moving at a constant velocity initially that starts uniformly accelerated motion at some moment. The EM field outside the future light sphere of the starting event is still the original Coulomb field, as if the charge were still moving uniformly. Physically, this Coulomb field becomes the δ-function field as the starting time $t \to -\infty$ [3,10]. The second difficulty is that the EM-field energy "near" a point charge is infinite. Moreover, it is hard to compare this infinite energy at one moment with that at another moment when the charge is moving at a speed different from the first moment. Fortunately, for a UAC, there are two moments for each speed at which the velocity of the charge at one moment is the negative of that at the other moment. Therefore, we may compare the total EM-field energy at one moment with that at the other moment and see whether energy is conserved.

II. EM FIELDS OF A UAC

The trajectory of a one-dimensional UAC can be written as $x = X$, $y = Y$, and $z = Z$, with $X = Y = 0$, $Z = (\alpha^2 + t^2)^{1/2}$, for a certain Lorentz frame (t,x,y,z), where α is a constant. We have set the speed of light $c = 1$. The radiation power calculated by the Larmor formula is

$$P = \tfrac{2}{3} e^2 a^\mu a_\mu = \frac{2e^2}{3\alpha^2} , \tag{1}$$

where e is the magnitude of the charge, and a^μ is the four-acceleration. We see that P is a constant. The EM field of a UAC can be found as [1,4,8,5]

$$\begin{aligned} &E_\phi = B_\rho = B_z = 0 , \\ &E_z = -4e\alpha^2 (Z^2 + \rho^2 - z^2)/\xi^3 , \\ &E_\rho = 8e\alpha^2 \rho z/\xi^3 , \\ &B_\phi = 8e\alpha^2 \rho t/\xi^3 , \end{aligned} \tag{2}$$

where

$$\xi=[(Z^2-\rho^2-z^2)^2+4\alpha^2\rho^2]^{1/2} ,$$

with (ρ,ϕ,z) being cylindrical coordinates. This field is valid only for $z+t>0$. To get a solution that is valid for the whole space-time, we need to multiply a step function $\theta(t+z)$ to the field in (2) and add (subtract) a δ-function term $2e\rho\delta(z+t)/[\rho^2+\alpha^2]$ to (from) the E_ρ (B_ϕ) term. The EM field given by (2) tends to that of a uniformly moving charge with constant velocity $v=t/Z$ in the region near the charge, i.e., $(z-Z)^2+\rho^2\ll\alpha^2$:

$$\begin{aligned} &E_\phi=B_\rho=B_z=0 ,\\ &E_z=e\gamma(z-Z)/[\gamma^2(z-Z)^2+\rho^2]^{3/2} ,\\ &E_\rho=e\gamma\rho/[\gamma^2(z-Z)^2+\rho^2]^{3/2} ,\\ &B_\phi=e\gamma\rho v/[\gamma^2(z-Z)^2+\rho^2]^{3/2} ,\end{aligned} \tag{3}$$

where $\gamma=(1-v^2)^{-1/2}=Z/\alpha$ is the relativistic factor.

III. COMPARISON OF TOTAL EM-FIELD ENERGY

Let us consider a charge originally moving at a constant velocity that starts uniformly accelerated motion. Its trajectory can be written as $X=Y=0$,

$$Z=\begin{cases} Z_1-t_1(t+t_1)/Z_1 & \text{for } t\le -t_1\\ (\alpha^2+t^2)^{1/2} & \text{for } t>-t_1 ,\end{cases} \tag{4}$$

where $Z_1\equiv(\alpha^2+t_1^2)^{1/2}$, and t_1 is assumed positive. Consider the EM field at a later time $t=-t_2<0$. Inside the light sphere centered at $z=Z_1$ with radius t_1-t_2, the EM field is given by (2). Outside this sphere, the EM field is given by (3) with $Z=Z_1$ and $z\to z+t_1(t_1-t_2)/Z_1$, since the charge would be at the position $z=Z_1-t_1(t_1-t_2)/Z_1$ if it were still moving at a constant velocity. Similarly, the EM field for time $t=t_2$ can be found, except that the radius of the light sphere is t_1+t_2 and $z\to z+t_1(t_1+t_2)/Z_1$ in (3) now. We can then calculate the difference in the total EM-field energy between $t=-t_2$ and $t=t_2$ [14]:

$$\Delta W=\frac{1}{8\pi}\left[\int d^3x(E^2+B^2)_{t_2}-\int d^3x(E^2+B^2)_{-t_2}\right] .$$

We may write

$$\Delta W=\Delta W_v+\Delta W_a , \tag{5}$$

where ΔW_v is the difference in the Coulomb-field energy and ΔW_a is the difference in the UAC field energy. By (3), it can be shown that

$$\Delta W_v=\frac{e^2\alpha^4}{4}\int_{-1}^{1}d\xi\int_{l_+}^{l_-}\frac{dl}{l^2}\frac{Z_1^2\xi^2+(2Z_1^2-\alpha^2)(1-\xi^2)}{[Z_1^2\xi^2+\alpha^2(1-\xi^2)]^3} , \tag{6}$$

where $l_\pm$ describe two light spheres:

$$l_\pm=(t_1\pm t_2)[t_1\xi+(\alpha^2+t_1^2\xi^2)^{1/2}]/Z_1 .$$

Note that we have used coordinates

$$l=[\rho^2+(z-Z_\pm)^2]^{1/2} ,\quad \xi=(z-Z_\pm)/l ,$$

with $Z_\pm=Z_1-t_1(t_1\pm t_2)/Z_1$. This can be written in dimensionless variables for the convenience of numerical calculation:

$$\Delta W_v=-W_L\frac{3\tau_2\gamma_1}{8\tau_1(\tau_1^2-\tau_2^2)}\times\int_{-1}^{1}\frac{d\xi[\gamma_1^2+\tau_1^2(1-\xi^2)]}{[\tau_1\xi+(1+\tau_1^2\xi^2)^{1/2}][1+\tau_1^2\xi^2]^3} , \tag{7}$$

where $\tau_1\equiv t_1/\alpha$, $\tau_2\equiv t_2/\alpha$, $\gamma\equiv(1+\tau_1^2)^{1/2}$, and

$$W_L\equiv 2t_1P=\frac{2e^2}{3\alpha^2}2t_1 , \tag{8}$$

with P given by (1). Similarly, ΔW_a can be found by (2):

$$\Delta W_a=4e^2\alpha^4\int_{-1}^{1}d\eta\int_{r_-}^{r_+}\frac{dr}{r^2}w(Z_2) , \tag{9}$$

where

$$w(Z)\equiv\frac{r^2+4Z\eta r+4[Z^2\eta^2+(2Z^2-\alpha^2)(1-\eta^2)]}{\{r^2+4Z\eta r+4[Z^2\eta^2+\alpha^2(1-\eta^2)]\}^3} ,$$

$$r_\pm=(Z_1-Z_2)\eta+[(t_1\pm t_2)^2-(Z_1-Z_2)^2(1-\eta^2)]^{1/2} ,$$

with $Z_2\equiv(\alpha^2+t_2^2)^{1/2}$. Note that we have used coordinates

$$r=[\rho^2+(z-Z_2)^2]^{1/2} ,\quad \eta=(z-Z_2)/r .$$

In dimensionless variables

$$\Delta W_a=W_L\frac{3}{\tau_1}\int_{-1}^{1}d\eta\int_{\lambda_-}^{\lambda_+}\frac{d\lambda}{\lambda^2}\frac{\lambda^2+4\gamma_2\eta\lambda+4[\gamma_2^2+\tau_2^2(1-\eta^2)]}{[\lambda^2+4\gamma_2\eta\lambda+4(1+\tau_2^2\eta^2)]^3} , \tag{10}$$

where $\lambda\equiv r/\alpha$, $\gamma_2\equiv(1+\tau_2^2)^{1/2}$. By (7) and (10), we can calculate ΔW numerically, with the λ integration in (10) done analytically first.

IV. NUMERICAL RESULTS

We used the Romberg method with equal step sizes to do the numerical integration. This may not be the best way to handle these kinds of integrations, but it is easy to program. Even with such an elementary method, it was found that $\Delta W=0$ for a very large range of parameter space (τ_1,τ_2). Only for τ_1 large or τ_2 very close to τ_1, could we not get accurate results because of numerical difficulties. These numerical difficulties mainly come from round-off errors due to cancellations between large numbers. We used up to quadruple precision (28 digits) in some calculations in order to get accurate results. Physical reasons for these difficulties are that, when τ_1 is large, $\tau_1 \approx \gamma_1$ (which is the relativistic factor), the charge is highly relativistic and the Coulomb field is large at the midplane ($\zeta \approx 0$); and when τ_2 is close to τ_1, both (7) and (10) include a large contribution from the "near" field, which is divergent. A few typical numerical results are shown in Table I. We only show the order of magnitude of the $|\Delta W/W_L|$ values, so that we can see more clearly how close to zero they are numerically. Note that they are not precise values, and will change if the number of integration steps changes. So, they actually indicate the precision of the numerical method for different parameters. For example, a value of 10^{-14} means that it is precise up to about 13 digits, and 10^{-4} means that it is precise up to about three digits, maybe 0.000 286, or 0.000 098 7, or whatever. In other words, those values converge to zero up to the precision of the numerical method allowed. They never converge to finite values. We have done calculations for many other cases within the range of parameters of Table I. The results are similar. From these results, although there are numerical difficulties as mentioned above, we feel confident that $\Delta W = 0$ at least for $\tau_1 \lesssim 100$, which is already highly relativistic. In order to extend this range, we need to run the program with higher precision, or we need to do some nontrivial analytical work and use a more sophisticated numerical method. However, the evidence given by the current results using the simple method is already so strong that we expect (7) and (10) do add up to zero identically for all positive τ_1, τ_2 with $\tau_2 < \tau_1$. To prove this and two other "identities" described below are well-defined mathematical problems. It is more worthwhile to try to prove them analytically than to try to strengthen this already strong, but never conclusive, numerical evidence. We are not able to prove them right now, but hopefully this can be done later.

V. RADIATION REACTION

Another reason why we think the above result is correct is that it is consistent with the fact that the radiation reaction [13],

$$\Gamma^\mu = \tfrac{2}{3} e^2 \left[\frac{da^\mu}{d\tau} - u^\mu a_\nu a^\nu \right] , \qquad (11)$$

TABLE I. Total EM-field-energy difference ΔW between $t = t_2$ and $t = -t_2$, for $t_2 < t_1$.

τ_1 $(=t_1/\alpha)$	τ_2 $(=t_2/\alpha)$	$\lvert\Delta W/W_L\rvert$
0.1	0.01	10^{-15}
0.1	0.099 999 999	10^{-11}
2	0.5	10^{-14}
2	1.999 999 99	10^{-12}
10	0.001	10^{-18}
10	1	10^{-15}
10	9.999 999 99	10^{-6}
100	0.1	10^{-11}
100	10	10^{-2}

vanishes for a UAC. In (11), u^μ is the four-velocity, and τ is the proper time. This consistence gives a strong support to the assumption that the total EM-field energy does remain constant when the radiation reaction vanishes, even though the Larmor formula gives a finite radiation power. Although it is not a conclusive proof, at least we did not find evidence showing that we need to introduce any non-EM "internal energy" in order to conserve energy within classical electrodynamics. The consistence of the EM-field-energy change with the radiation-reaction theory can also be shown in two other cases.

A. Energy change when uniformly accelerated motion starts

We may also compare the EM-field energy at $t = t_1$ with that of the original uniformly moving charge. This is equivalent to putting $t_2 = t_1$ in the previous case. We cannot use (7) and (10) directly to calculate ΔW for this case. ΔW is still given by (5), (6), and (9), with $t_2 = t_1$. However, both (6) and (9) are divergent now. We need to combine the two and cancel the near-field contribution analytically. One way to do so is to introduce a change of variables in (6):

$$\begin{aligned} l &= r\Lambda , \\ \zeta &= \frac{1}{\Lambda} \left[\eta + \frac{r}{2Z_1} \right] , \\ \Lambda &= \left[1 + \frac{\eta r}{Z_1} + \frac{r^2}{4Z_1^2} \right]^{1/2} . \end{aligned} \qquad (12)$$

This transformation does not change the contribution from the "near field" to the EM-field energy since $r \to l$, $\eta \to \zeta$ as $r \to 0$. Then ΔW_v becomes

$$\Delta W_v = -4e^2\alpha^2 \int_0^{2t_1} \frac{dr}{r^2} \int_{-1}^{1} d\eta \left[1 + \frac{r\eta}{Z_1} \right] w(Z_1) .$$

By this, (5), and (9), ΔW can be expressed in dimensionless variables as

$$\Delta W = W_L \frac{3}{16v} \int_0^v dy \left[\left[\frac{1}{\gamma_1^2} - 2 - y^2 \right] \left[\frac{J_{13}}{y} \right] - 2J_{23} + \left[1 - \frac{1}{\gamma_1^2} \right] \left[\frac{J_{33}}{y} \right] \right] , \qquad (13)$$

where

$$\begin{aligned} J_{mn} &\equiv \int_{-1}^{1} d\eta \frac{\eta^m}{(a\eta^2 + b\eta + c)^n} , \\ v &\equiv \tau_1/\gamma_1 , \quad y \equiv r/2Z_1 , \\ a &\equiv \tau_1^2 , \quad b \equiv 2\gamma_1^2 y , \quad c \equiv \gamma_1^2 y^2 + 1 . \end{aligned}$$

Note that J_{mn} can be found by analytical integration, and that J_{13}/y and J_{33}/y are finite as $y \to 0$. By (13), it was found numerically that $\Delta W = W_L/2$ for a very large range of τ_1. Actually, this can be shown analytically for $\tau_1 \to 0$. Some typical numerical results are shown in Table II. The numerical method and interpretation of results are similar to those discussed above. We also did calculations based on a variables transformation different from (12) and got similar results. We expect that (13) is actually an identity, $\Delta W = W_L/2$ for all τ_1, although we cannot prove this analytically.

We see that there is a sudden jump of ΔW from 0 for $t_2 < t_1$ to $W_L/2$ for $t_2 = t_1$. The only reasonable explanation is that there is a sudden increase in the EM-field energy when the uniformly moving charge starts moving in uniformly accelerated motion, since there is also a sudden change in acceleration at that time. This sudden increase in energy can be explained by the work done against the radiation reaction (11) at that moment. It can be shown by (4) and (11) that

$$\Gamma^0 = -\frac{\gamma}{2} W_L \delta(t+t_1) ,$$

where $\gamma = Z_1/\alpha$ is the relativistic factor. This means that the external force that starts the uniformly accelerated motion must supply an energy $W_L/2$ at the starting moment, and this energy becomes EM-field energy immediately.

B. Energy change when uniformly accelerated motion ends

If we now let the charge change back to uniform motion, i.e., $Z = Z_1 + t_1(t - t_1)/Z_1$, for $t > t_1$, it is also possible to compare the EM-field energy at $t = t_2 > t_1$ with that at $t = -t_1$. The calculation is very similar to the case $t_2 < t_1$. We still calculate ΔW by (5), with ΔW_a given by (10) for $\tau_2 > \tau_1$ and ΔW_v given by the right-hand side of (7) multiplied by τ_1/τ_2. Numerically, we found that $\Delta W = W_L$ for a very large range of parameter space (τ_1, τ_2). Some typical numerical results are shown in Table III. This result is expected, since as the charge changes back to uniform motion, the total increase in the EM-field energy should be equal to that found by the Larmor formula, i.e., W_L defined by (8). We see that the EM-field energy also has a sudden increase at $t = t_1$, since $\Delta W = W_L/2$ for $t_2 = t_1$ but $\Delta W = W_L$ for $t_2 > t_1$. This sudden increase in energy can also be explained by the work done against the radiation reaction. By (11), we now have

$$\Gamma^0 = -\frac{\gamma}{2} W_L [\delta(t-t_1) + \delta(t+t_1)] .$$

This means that the external force has to supply an energy $W_L/2$ at $t = t_1$ in order to bring the charge back to uniform motion immediately.

TABLE II. Total EM-field-energy difference ΔW between $t = t_1$ and $t = -t_1$.

τ_1 $(=t_1/\alpha)$	$\lvert \Delta W/W_L - 0.5 \rvert$
10^{-2}	10^{-14}
1	10^{-14}
20	10^{-14}
40	10^{-11}
100	10^{-5}

TABLE III. Total EM-field-energy difference ΔW between $t = t_2$ and $t = -t_1$, for $t_2 > t_1$.

τ_1 $(=t_1/\alpha)$	τ_2 $(=t_2/\alpha)$	$\lvert \Delta W/W_L - 1 \rvert$
0.1	0.100 000 2	10^{-8}
0.1	2	10^{-14}
0.1	2×10^6	10^{-13}
1	1.000 02	10^{-11}
1	2×10^5	10^{-14}
10	10.000 000 002	10^{-6}
10	20	10^{-8}
10	100	10^{-6}
30	30.002	10^{-4}
30	300	10^{-2}

VI. DISCUSSION

The above three numerical results show directly that the radiation-reaction formula (11) is consistent with energy conservation of a UAC. Return to the question whether a UAC radiates. If we believe that the total EM-field energy of a charge does remain constant during its uniformly accelerated motion, as suggested by the above numerical results, the question whether the charge radiates becomes a matter of definition. If we define radiation by the increase of the EM-field energy, a UAC does not radiate. If we define radiation by the power that flows out of the future light sphere of a certain retarded point, i.e., by means of the Larmor formula, a UAC radiates all the time, with the constant emission power given by (1). It is actually not necessary to explain energy conservation for the latter definition since energy is always conserved. Energy simply flows from one region of space to another.

There is a problem with the explanation that radiation energy comes from the EM-field energy "near" the UAC. We see that the "near" field energy is exactly the same at two moments $t = -t_2$ and $t = t_2$. The increase in energy of the UAC field can only come from the decrease in energy of the Coulomb field outside the UAC field region. This is expected by the explanation that radiation energy comes from the δ-function field, which is physically a Coulomb field.

However, there is a problem with the explanation by the δ-function field also. As $t \to \infty$, the Coulomb-field energy outside the UAC field region tends to zero. It is not possible to supply a constant power to the UAC field. The only source now must be the "near" field of the charge. So we see that both explanations have their own validity and limitation.

Perhaps these different points of view will not seem so contradictory if we note that no realistic charged particle can undergo uniformly accelerated motion forever. Radiation energy must ultimately be supplied by the work done against the radiation reaction. Then both definitions above will give the same conclusion that the charge does radiate.

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